Nonlinearities in Hydrodynamics:

Is Israel-Stewart hydro self-consistent?

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- What is Relativistic Hydro?
- Why we need to do second order hydro (Israel-Stewart)
- Self-consistency: hydro's contribution to hydro coeff.
- Limits on Viscosity and the Problem Defining τ_π
- Conclusions

What is a fluid?

Many degrees of freedom

No local spont. breaking of trans, rotat. invariance

When does it behave hydrodynamically?

Define “mesoscopic” regions: each has many DOF, but is nearly uniform.

Mesoscopic region near internal equilibrium (thermodynamics applicable on meso scale)

Are there interesting *relativistic* examples?

Sure!

- Early universe cosmology: Universe was a relativistic fluid through most of its evolution. Applications: baryogenesis, leptogenesis, *etc*

- Heavy ion collisions: studied in the laboratory!

- Possibly laboratory E&M plasmas (intense laser program)

Most current interest in heavy ion collisions

Ideal Hydrodynamics

Ideal hydro: stress-energy conservation

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{(4 equations, 10 unknowns)}$$

plus local equilibrium *assumption*:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} = \epsilon u^\mu u^\nu + P(\epsilon)\Delta^{\mu\nu}, \\ u^\mu u_\mu &= -1, \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \end{aligned}$$

depends on 4 parameters (ϵ, 3 comp of u^μ): closed.

What are systematic corrections?

Nonideal Hydro

Assume that ideal hydro is "good starting point," look for small systematic corrections.

Near equilibrium iff $t_{\rm therm} \ll t_{\rm vary}, l_{\rm vary}/v$ (so ∂ small)

Allows expansion of corrections in gradients:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu}[\partial,\epsilon,u] \\ \Pi^{\mu\nu} &= \mathcal{O}(\partial u, \partial\epsilon) + \mathcal{O}(\partial^2 u, (\partial u)^2, \ldots) + \mathcal{O}(\partial^3 \ldots) \end{aligned}$$

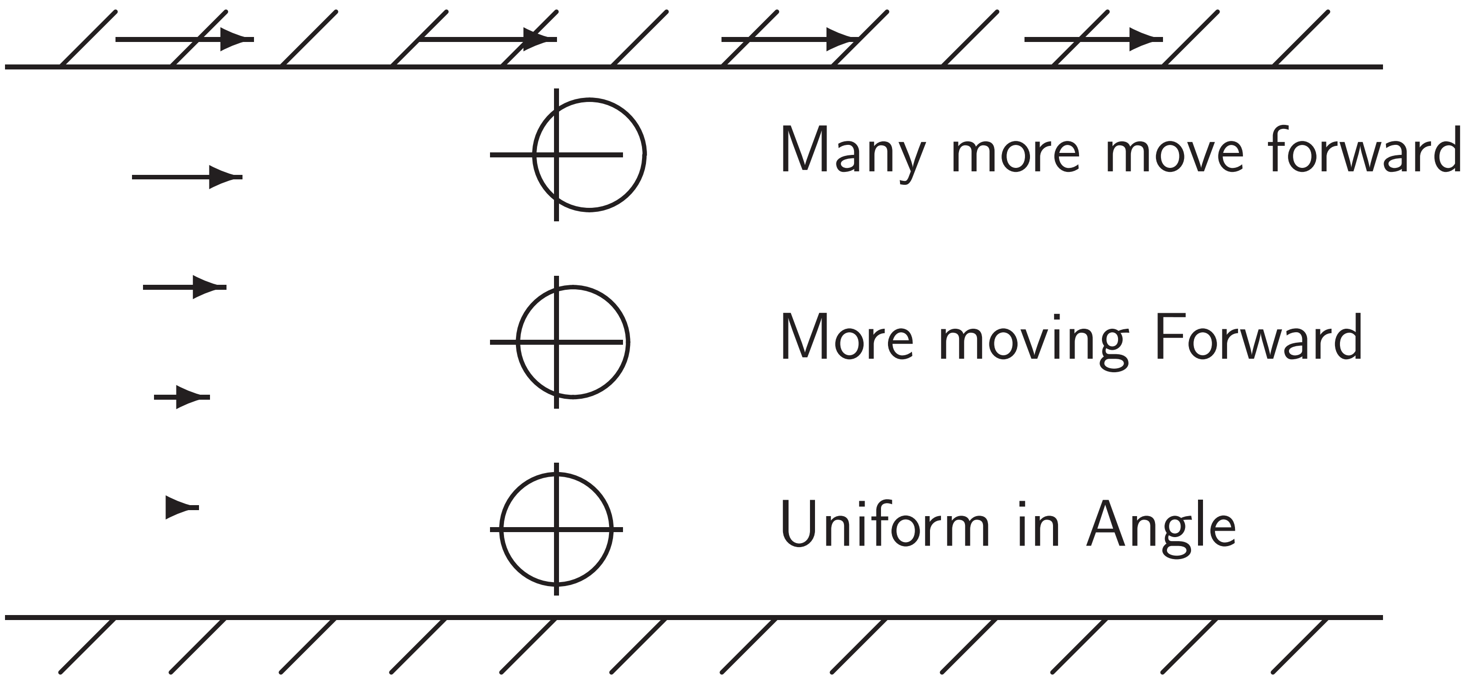
For Conformal theory $T^\mu_\mu = 0 = \Pi^\mu_\mu$, 1-order term unique:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}\,, \quad \sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta}\partial\cdot u\right)$$

Coefficient η is shear viscosity.

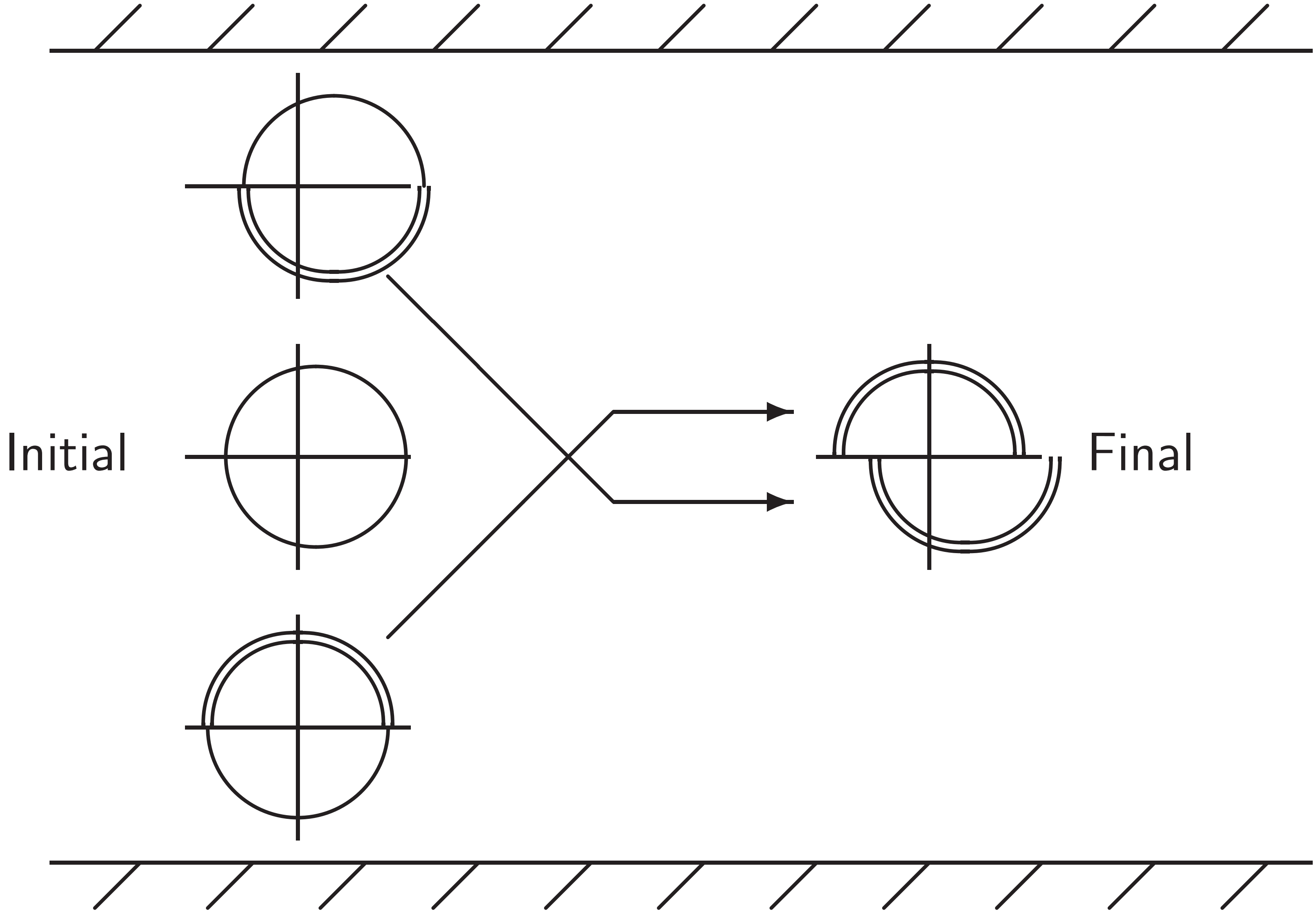
What is shear viscosity?

Consider non-uniform fluid flow of *particles*:



(Plotting ϕ=direction, $r = \#$ particles in that ϕ)

Propagation by Mean Free Path



Skewed momentum distribution.

Viscosity as transverse momentum diffusion

Small-fluctuation analysis of (Navier-Stokes) $\partial_\mu T^{\mu\nu} = 0$
Plane wave varying in z, t, consider $\partial_\mu T^{\mu x}$:

$$\partial_\mu \Big((\epsilon + P) u^\mu u^x + P g^{\mu x} - \eta \Delta^{\mu\alpha} \partial_\alpha u^x \Big) = 0$$

To linear order in spatial u, this is

$$(\epsilon + P) \partial_t u^x - \eta \partial_z^2 u^x = 0$$

Diffusion equation for u^x: $\eta/(\epsilon{+}P)$ is diffusion coefficient.

Propagation speed of information is infinite!

Is infinite propagation speed a problem?

Argument against: smooth initial data

If heat equation initial data smooth, propagation speed is finite.

Only believe hydro for smooth conditions. No problem?

Argument for: numerical reality

Nonlinear equations: non-smooth stuff always gets excited.

Do numerical work with coarse space-time lattice: UV modes not present. But discretization errors "numerical viscosity".

If I really want some given η value, I *have* to over-resolve!

Israel-Stewart approach

Add one second order term:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_\pi \, u^\alpha \partial_\alpha \sigma^{\mu\nu}$$

Make (1'st order accurate) $\eta\sigma \longrightarrow -\Pi$ in order-2 term:

$$\tau_\pi \, u^\alpha \partial_\alpha \Pi^{\mu\nu} \equiv \tau_\pi \, \dot{\Pi}^{\mu\nu} = -\eta\sigma^{\mu\nu} - \Pi^{\mu\nu}$$

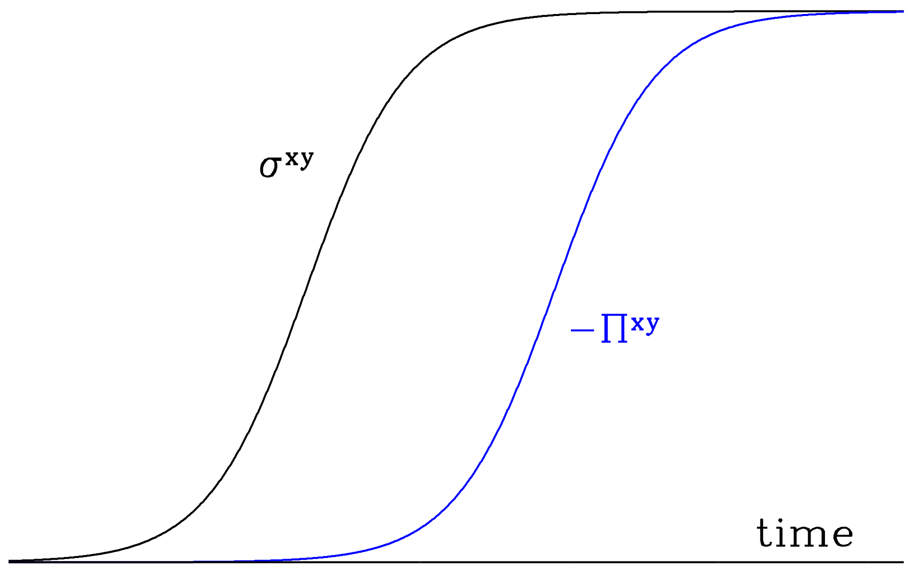
Relaxation eq driving $\Pi^{\mu\nu}$ towards $-\eta\sigma^{\mu\nu}$.

Momentum diff. no longer instantaneous.

Causality, stability are restored (depending on τ_π)

Viscous hydro for heavy ions all use this!

How to think about τ_π



Suppose shear flow σ^{xy} "turns on" as a function of time. Then Π^{xy} rises to $-\eta\sigma^{xy}$. But there is a *delay* in Π^{xy} turning on. τ_π is average length of the delay.

Time for flow pattern in slide 7 to develop.

Full 2'nd order hydro

Assume *conformality* and *vanishing chem. potentials*:

5 possible terms Baier *et al*, [arXiv:0712.2451]

$$\Pi^{\mu\nu}_{2\text{ ord.}} = \eta\tau_\pi \left[u^\alpha \partial_\alpha \sigma^{\mu\nu} + \frac{1}{3}\sigma^{\mu\nu}\partial_\alpha u^\alpha \right] + \lambda_1 \left[\sigma^\mu_\alpha \sigma^{\nu\alpha} - (\text{trace})\right]$$
$$+ \lambda_2 \left[\frac{1}{2}(\sigma^\mu_\alpha \Omega^{\nu\alpha} + \sigma^\nu_\alpha \Omega^{\mu\alpha}) - (\text{trace}) \right]$$
$$+ \lambda_3 \left[\Omega^\mu{}_\alpha \Omega^{\nu\alpha} - (\text{trace})\right] + \kappa \left(R^{\mu\nu} - \ldots\right) \, ,$$
$$\Omega_{\mu\nu} \equiv \frac{1}{2}\Delta_{\mu\alpha}\Delta_{\nu\beta}(\partial^\alpha u^\beta - \partial^\beta u^\alpha) \;\; [\text{vorticity}] \, .$$

What are η, τ_π?

Are they parameters in an *ad hoc*, unjustified model?

Are they numerical coefficients in a rigorously defined effective theory?

Are they *scale dependent* Wilson coefficients in an effective field theory?

Hint: Kubo relation Baier *et al*, arXiv:0712.2451 [hep-th]

$$G_{ra}^{xy,xy}(\omega) = P - i\eta\omega + \eta\tau_\pi\omega^2 + \ldots$$

Why you should worry

We said $\Pi^{\mu\nu} = \mathcal{O}(\partial u) + \mathcal{O}(\partial^2 u, (\partial u)^2) + \ldots$

based on assumption thermalization is local, microscopic.

Hydro itself predicts long-lived shear,sound modes:

$$0 = \partial_\mu \Big(T^{\mu\nu} = (\epsilon{+}P)u^\mu u^\nu + P g^{\mu\nu} - \eta\sigma^{\mu\nu} \Big)$$

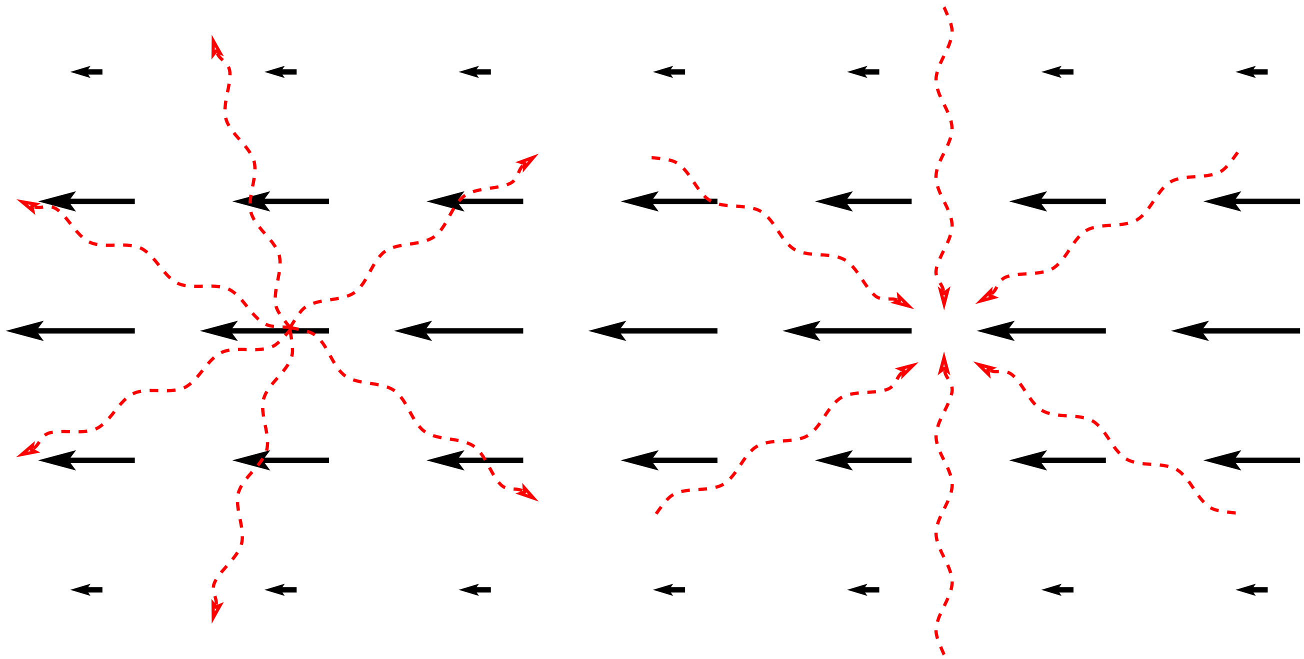
fluctuations in u^μ, ϵ: dispersion relations

$$\omega_{\rm shear} = i\frac{\eta}{\epsilon + P}k^2 \,, \qquad \omega_{\rm sound} = \pm\frac{k}{\sqrt{3}} + i\frac{2\eta}{3(\epsilon + P)}k^2$$

Small k: long lived, dissipation *not* local,microscopic

Hydro Waves Contribute to Viscosity!

Consider shear flow:



Flow decays because x-momentum leaves (diffuses from) flowing region. One mechanism: propagation of hydro (sound) waves!

Naive estimate: sound waves as "particles"

"stumbling drunk" with step length λ, time between steps $t = \lambda/c$: diffusion constant is $\lambda^2/t \sim c\lambda$.

Mean free path of a sound wave $\lambda \sim (\epsilon + P)/(\eta k^2)$.

Equipartition: Energy $T/2$, Mom. $T/2c$ per sound mode

Contribution of sound modes to momentum diffusion

$$\eta_{\text{from soundwaves}} \sim \int d^3k\, T\lambda \sim \int d^3k \frac{T(\epsilon + P)}{k^2 \eta}$$

Finite. Time scale to establish this contribution:

$$\eta\tau_\pi \sim \int d^3k\, T\lambda t \sim \int d^3k \frac{T(\epsilon + P)^2}{k^4 \eta^2} \qquad \text{Divergent!!}$$

How to compute hydro contribution to η, τ_π

Use Kubo relation

$$G_{ra}^{xy,xy}(\omega) = P - i\eta\omega + \eta\tau_\pi\omega^2 + \ldots$$

Calculate contrib. of hydro modes themselves to G^{xyxy}.

$$\text{Feynman rules:} \quad T^{ij} = (\epsilon + P)u^i u^j + P g^{ij}\,,$$

$$\langle u^i u^j(k,\omega)\rangle = \frac{T}{\epsilon + P}\frac{(\delta^{ij} - \hat{k}^i\hat{k}^j)2\gamma_\eta k^2}{(\gamma_\eta k^2 - i\omega)(\gamma_\eta k^2 + i\omega)}\text{shearwave}$$

$$\left[\gamma_\eta = \tfrac{\eta}{\epsilon+P}, \gamma'_\eta = \tfrac{4}{3}\gamma_\eta\right] \qquad + \frac{T}{\epsilon + P}\frac{(\hat{k}^i\hat{k}^j)2\gamma'_\eta k^2\omega^2}{(\omega^2 - k^2/3)^2 + (\gamma'_\eta k^2\omega)^2}\text{soundwave}$$

Think of hydro as IR effective theory, η etc are Wilson coeff.

Computing $G_{ra}^{xy,xy}(\omega, k=0)$

Straightforward application of Feynman rules:

$$G_{ra}^{xy,xy}(\omega)[\text{hydro}] = -i\omega \left(\frac{17 T k_{\text{max}}}{120\pi^2 \gamma_\eta} \right) + (i+1)\omega^{\frac{3}{2}} \frac{7 + \left(\frac{3}{2}\right)^{\frac{3}{2}} T}{240\pi \gamma_\eta^{3/2}}$$

k_{max}: k-scale above which hydro incorrect/inconsistent.

- $-i\omega$ term: extra contrib. to η
- $i\omega^{3/2}$: effective ω dependence of η.
- $\omega^{3/2}$: like τ_π but *wrong* ω dependence.

Lesson: η

Small η: freer propagation of sound, shear modes.
More efficient momentum transport, raising η.

Depends on $k_{\rm max}$. Where does hydro break down?
Scale where it's no longer self-consistent.

Safe guess: $k_{\rm max} < \tau_\pi^{-1}/2$. In $\mathcal{N}$=4 SYM, this is about $2T$.

- $\mathcal{N}$=4SYM: added η/s is $\sim 1/N_{\rm c}^2$.
- Weak coupling: $\eta_{\rm from\ hydro} \sim \alpha^4$ while $\eta_{\rm tot} \sim \alpha^{-2}$
- Real QCD: $\frac{\eta}{s} = .16$: add 0.01. $\frac{\eta}{s} = .08$: add 0.036!

Lesson: τ_π

Formally, τ_π does not exist! (At least it's not a number)

But 2-order hydro can make sense as effective theory over some range of frequencies if, in that range,

$$G^{xyxy} \sim -i\omega\eta + (i+1)\omega^{3/2} \frac{\left[7 + \left(\frac{3}{2}\right)^{\frac{3}{2}}\right] T(\epsilon{+}P)^{\frac{3}{2}}}{240\pi\eta^{\frac{3}{2}}} + \omega^2\eta\tau_\pi$$

has the $\omega^{3/2}$ term sub-dominant. Requires:

$$\frac{T^2(\epsilon{+}P)^3}{(4\pi)^2\eta^3} \ll \eta^2\tau_\pi$$

Then, in this range, τ_π is nearly ω and scheme-independent Wilson coefficient

When does this range exist? Weak coupling:

$$\eta \sim \alpha^{-2} T^3 \,, \quad \tau_\pi \sim \alpha^{-2} T^{-1} \,, \quad \text{OK if } \alpha^2 T \gg \omega \gg \alpha^{14} T$$

Large $N_{\rm c}$ (many degrees of freedom):

$$\eta \sim N_{\rm c}^2 T^3 \,, \quad \tau_\pi \sim N_{\rm c}^0 T^{-1} \,, \quad \text{OK if } T \gg \omega \gg N_{\rm c}^{-4} T \,.$$

Real-world QCD at $T = 200\text{MeV}$: $(\epsilon + P) \sim 10 T^4$,

- if $\eta/s = 0.16$, 2-order hydro valid for $T \gg \omega \gg T/20$
- if $\eta/s = 0.08$, 2-order hydro valid for $2T \gg \omega \gg 2.8T$

Conclusions

- Relativistic hydrodynamics has interesting applications
- Need 2'nd order Hydro for stability (Thanks Werner!)
- Hydro waves contribute to hydro coefficients!
- Hydro contributions to η means it cannot be too small
- No absolute definition for τ_π. Wilson coeff.
- Range of validity of 2-order hydro: wide unless η very small (but possibly an issue for heavy-ion collisions!)